The Vehicle Routing Problem

A brief summary of the famous combinatorial optimization problem

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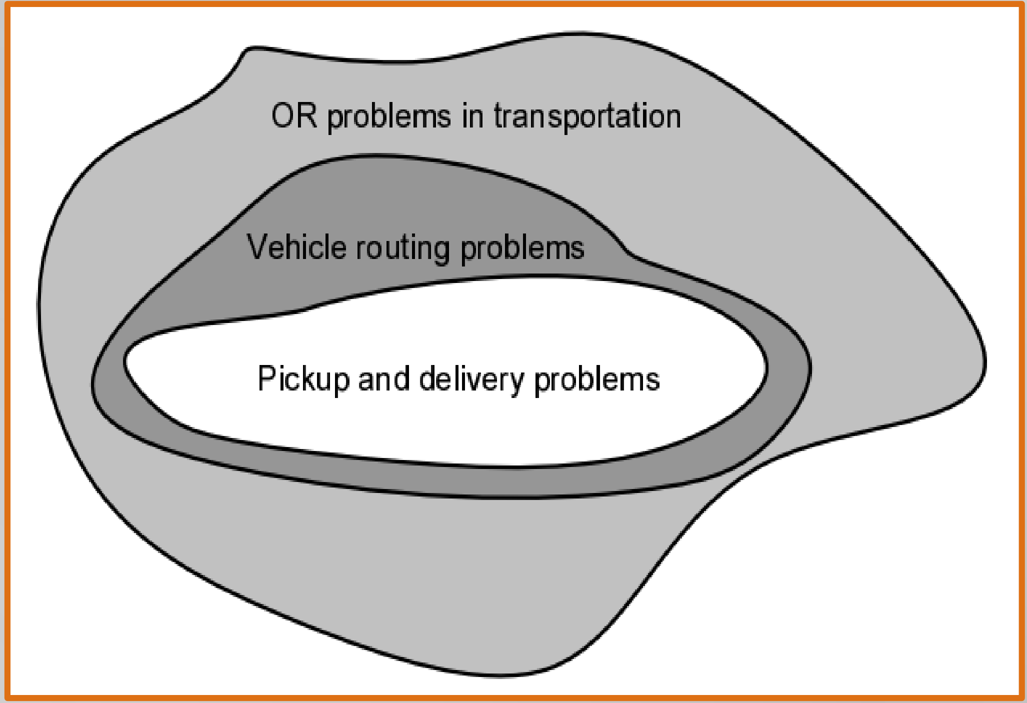
Overview

- History
- Basics of VRP
- A Mathematical Model
- The Subtour Constraint
- Variations
- Methods



History

The classical Vehicle Routing Problem (VRP) was first proposed as a linear program in 1958 by Dantzig and Ramser during the peak of Business Science development (Operations Research) [3]. They wanted to find an optimal solution to delivering gasoline to various service stations. Their algorithmic solutions are explained in the article published in the late fifties titled, *The Truck Dispatching Problem.* The discipline of Operations Research, which housed many mixed integer problems like the VRP, quickly become recognized as a subject of high interest.

Introduction to VRP

The VRP is essentially an extension of a much simplifier problem, the Traveling Salesperson Problem (TSP). In its' trivial from, the problem aims to determine the shortest route passing through each *n* points or *nodes* (locations) exactly once.

<u>VRP model</u>

- *m* number of vehicles in operation, located initially at a warehouse *{0}*.
- *n* number of customers to be visited exactly once.
- Vehicles deliver products to designated customers throughout a network and return to the initial warehouse at the end of the period.
- There is an associated cost *c* with each route.
- The solution to the classical VRP is the set of routes assigned to drivers that satisfy all constraint conditions while minimizing transportation cost.

A Mathematical Model

V is the set of nodes {0,1,2…..n} where 0 is the central depot and an edge is defined as the pair *(i,j)*. c_{ij} is the cost of transportation for the associated edge. x_{ij} is defined as 1 if the edge (i,j) is selected as a route, and 0 otherwise. Many models assume that $i < j$ for the sake of symmetry. m is the number of vehicles or workers. [4]

Minimize : $\sum_{(i,j)\in E} c_{ij}\, x_{ij}$

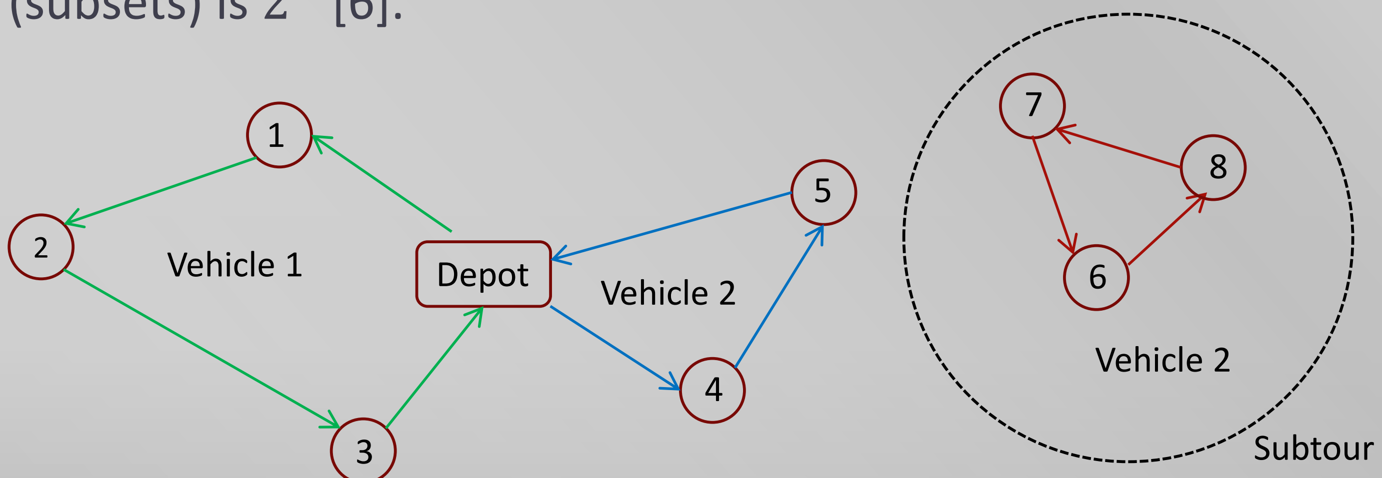
Subject to : $\sum_{(j:i<j)} x_{ij} + \sum_{(j:j<i)} x_{ji} = 2, \quad i \in V - \{0\}$

and $\sum_{j\in V-\{0\}} x_{0j} = 2m$

Given that : $x_{ij} \in \{0,1\}$ $\;and\, (i,j) \in E$

Subtours in VRP

Given the previous mathematical model a valid (and optimal) solution may include a *subtour*. The idea is that over a network our LP was solved in a way such that paths were connected to create "mini-routes" rather than one large one. The cardinality of the set containing all possible subtours grows exponentially with the amount of nodes as the total number of all possible subtours (subsets) is 2^n [6].

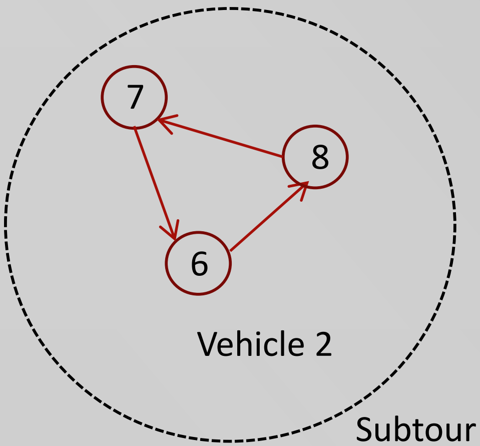


Subtour Constraint

Identification

For a particular subset *S*

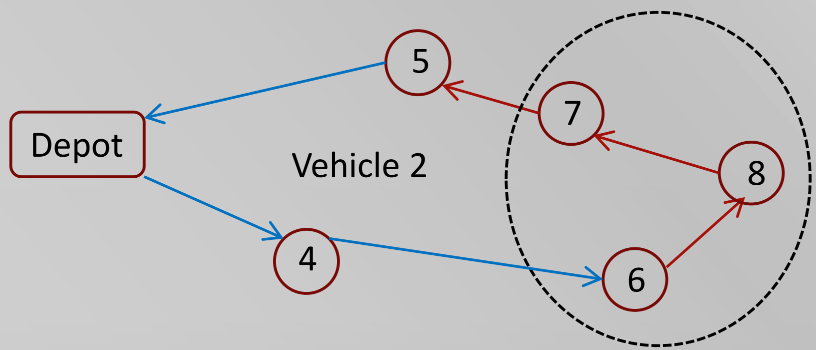
$\sum x_{ij} = |S|$,where S⊂V-{0}



Elimination

$$\sum_{\substack{i,j\in S \\ i<j}} x_{ij} \le |S| - 1$$

Where S⊂V-{0}



Variations of Classical VRP

- **Capacitated VRP (CVRP)** : Each vehicle has a specific max capacity
- **VRP with Time –Windows (VRPTW)** : Visits to nodes must occur during a certain time-window.
- **VRP with Pickup and Delivery (VRPPD)** : Each node has an associated *demand* value (often represented in binary).
- **VRP with Multiple Trips (VRPMT)** : Contrary to the classical model, each vehicle has the option of running more than one route.
- **Open VRP (OVRP)** : Vehicles are not required to return to the central depot
- **VRP with Backhauls (VRPWB)** : Customers can requested a return or exchange
- **Multiple Depot VRP(MDVRP)** : No longer one single depot, but multiple throughout network.

Solution Methods

- Exact methods – Global optimality is guaranteed yet often impractical in terms of time and space.
- *Heuristics* – "Quickly" finds a feasible solution yet has no guarantee of optimality. Algorithms implement knowledge of the reality that brute force methods ignore (common sense, rule of thumb, etc.).

As the VRP is classified as NP-Hard problem, approximated solutions are considered desirable. For this reason alone, many consider heuristic methods to be a favored technique. [2]

Heuristics

Constructive – Solutions are built iterately based on previous partial solutions

- Nearest Neighbor Heuristic - (NNH)
- Cheapest Insertion Heuristic
- 2-phase Algorithm (clustering)

Local Searches – An initial feasible solution is proposed then algorithm seeks to improve this original solution.

- *K-opt (2-opt and 3-opt)*
- Descent algorithm

Metaheuristics

Heuristic methods are effective in local optimization yet often fail in the global setting. *Metaheuristics* uses similar techniques but adds an element of randomness or memory, in an attempt to avoid being “trapped” at a local optimal solution.

- Simulated Annealing (SA) – Uses a stochastic relation technique based on the *Boltzmann* probability distribution
- Tabu Search (TS) – Deterministic approach with a memory log.
- Genetic Algorithms – Considers sets of solutions simultaneously. Mimics biological process of evolution.

Exact Methods

As the sub tour constraint reveled the *NP –hardness* of the problem, exact methods cannot be solved in polynomial time. Realistically, these methods can only be used to solve small problems.

- Complete enumeration – simplest in design
- *Branch and bound* – partition solution and determine optimal solution for the subspace.
- *Branch and cut* using *cutting plane* techniques – Consider an additional constraint that may limit the feasible region but produce desirable (integer) results.

Summary

- The VRP is really an umbrella term for a class of dozens of other unique problems.
- The subtour elimination constraint adds a level of complexity, increasing computation time (no longer polynomial).
- Heuristics and metaheuristics are more reliable techniques than exact methods dues to the NP-Hardness of the VRP.

Sources

- Picture on title page retrieved from: http://proceedings.esri.com/library/userconf/proc97/proc97/to450/pap439/p439.htm

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